

SOME SPECTRAL PROPERTIES AND ISOPERIMETRIC INEQUALITIES FOR A NONLOCAL LAPLACIAN PROBLEM

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ABSTRACT. In this paper we consider a non-local problem for a Laplace operator in a multidimensional bounded symmetric domain. The investigated problem is an analogue of the classical periodic boundary value problems in the case of non-rectangular domain. We prove self-adjointness of the problem and show a method of constructing eigenfunctions. We obtain an analogue of the Rayleigh type inequality and some spectral inequalities for the first eigenvalue of the nonlocal problem.

1. INTRODUCTION

Eigenvalues of the Laplacian represent frequencies in wave motion, rates of decay in diffusion, and energy levels in quantum mechanics. Constructing eigenvalues is a difficult task: they are known in an explicit form only for some special domains. This fact leads to necessity of considerable work on estimating eigenvalues in terms of simpler geometric quantities such as area and perimeter.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, symmetric with respect to the origin and with a smooth boundary $\partial\Omega$. This symmetry means that alongside with a point (x_1, \dots, x_n) her "opposite" point $x^* = (-x_1, \dots, -x_n)$ also belongs to the domain. Let us denote $\partial\Omega_+ = \partial\Omega \cap \{x_1 \geq 0\}$, $\partial\Omega_- = \partial\Omega \cap \{x_1 < 0\}$.

Consider the following problem:

Problem P *Find a function satisfying the equation*

$$-\Delta u(x) = f(x), x \in \Omega, \quad (1.1)$$

in the domain Ω , and satisfying the following boundary conditions

$$u(x) = -u(x^*), x \in \partial\Omega_+, \quad (1.2)$$

$$\frac{\partial u(x)}{\partial n_x} = \frac{\partial u(x^*)}{\partial n_x}, x \in \partial\Omega_+. \quad (1.3)$$

Here n_x is a derivative in the direction of an outer normal to $\partial\Omega$.

Note that the problem P in the case of a disk, and a multidimensional ball was first formulated and investigated in [1]-[3].

The objectives of this work are:

- proving self-adjointness of the problem P;
- obtaining analogue of the Rayleigh type inequality for the first eigenvalue of problem P and of the inverse operator to problem P;
- constructing all the eigenfunctions of the problem P;

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- proving some spectral inequalities for the first and second eigenvalues of the problem P;
- getting estimates for the norm of the inverse operator to problem P.

Note that the isoperimetric and asymptotic inequalities for the eigenvalues of the Laplace operator and of the convolution type operators were obtained in [4]-[13].

Using the method of [2], we can prove

Theorem 1.1. *For any $f \in L_2(\Omega)$ a strong solution of problem P exists and is unique. This solution belongs to the class $W_2^2(\Omega)$ and it is represented as*

$$u(x) = \mathcal{L}_\Omega^{-1} f(x) = \int_{\Omega} G_P(x, y) f(y) dy, \quad (1.4)$$

where $G_P(x, y)$ is a Green's function of the problem P, which has the form:

$$G_P(x, y) = \frac{1}{2} [G_D(x, y) + G_D(x, y^*) + G_N(x, y) - G_N(x, y^*)]. \quad (1.5)$$

Here $G_D(x, y)$ is the Green's function of the Dirichlet problem and $G_N(x, y)$ is the Green's function of the Neumann problem.

It is well known that the Green's function of the Dirichlet problem in a unit ball of arbitrary dimension can be constructed by reflection method. Note that the Green's function of the Neumann problem in the unit ball of arbitrary dimension can be constructed explicitly [14]. Thus, the Green's function of the problem P in the unit ball of arbitrary dimension can be constructed explicitly.

2. MAIN RESULTS

Let $\lambda_1^D(\Omega)$ be a first eigenvalue of the Dirichlet problem, and $\lambda_2^N(\Omega)$ be a second eigenvalue of the Neumann problem in Ω . Also we denote by $\mu_1(\Omega)$ a first eigenvalue and by $\mu_2(\Omega)$ a second eigenvalue of the integral operator \mathcal{L}_Ω^{-1} .

Theorem 2.1. *Let $\lambda_1(\Omega)$ be a first eigenvalue of the problem P. Then*

$$\begin{aligned} \lambda_1(\Omega) &= \lambda_2^N(\Omega), \\ \lambda_1(\Omega) &< \lambda_1^D(\Omega). \end{aligned}$$

Theorem 2.2. *Let B be a ball of the same measure as Ω , i.e. $|B| = |\Omega|$. Then for the first eigenvalue of the problem P the Rayleigh type inequality:*

$$\lambda_1(\Omega) \leq \lambda_1(B) \quad (2.1)$$

is true.

That is, the ball maximizes the first eigenvalue of the problem P among all domains with equal measure.

By Theorem 2.2 the following results take place:

Corollary 2.3. *Let B be a ball of the same measure as Ω , i.e. $|B| = |\Omega|$. Then for the first eigenvalue of the operator \mathcal{L}_Ω^{-1} the following inequality:*

$$\mu_1(\Omega) \geq \mu_1(B)$$

is true.

That is, the ball minimizes the first eigenvalue of the operator \mathcal{L}_Ω^{-1} among all domains with equal measure.

Corollary 2.4. *Let D be a disk of the same measure as $\Omega \in \mathbb{R}^2$, i.e. $|D| = |\Omega|$. Then for the first and second eigenvalues of the operator \mathcal{L}_Ω^{-1} the following inequality:*

$$\mu_1(\Omega) + \mu_2(\Omega) \geq \mu_1(D) + \mu_2(D),$$

holds.

Corollary 2.5. *Let D be a disk with given diameter d of the same measure as $\Omega \in \mathbb{R}^2$, i.e. $|D| = |\Omega|$. Then the norm of the operator \mathcal{L}_Ω^{-1} is given by:*

$$\|\mathcal{L}_\Omega^{-1}\| \geq \frac{\pi p}{d},$$

where $p = 1.8412\dots$ is the first positive zero of the derivative of the Bessel function.

Results of Corollary 2.4 and Corollary 2.5 follow from Theorem Szegő [13].

Theorem 2.6. *Let B be a ball of the same measure as Ω , i.e. $|B| = |\Omega|$. Then the following inequalities hold:*

$$\frac{\lambda_1(\Omega)}{\lambda_1^D(\Omega)} \leq \frac{\lambda_1(B)}{\lambda_1^D(B)}, \quad (2.2)$$

$$\lambda_1^D(\Omega) - \lambda_1(\Omega) \geq \lambda_1^D(B) - \lambda_1(B). \quad (2.3)$$

The proofs of theorems are given in section 6. First of all, we consider the method of constructing eigenfunctions of the problem P.

3. OPERATOR PROPERTIES OF THE PROBLEM P

Theorem 3.1. *The problem P is self-adjoint in Hilbert space $L_2(\Omega)$.*

Proof. By \mathcal{L}_Ω we denote a closure in $L_2(\Omega)$ of the operator defined by the differential expression

$$\mathcal{L}_\Omega u = -\Delta u(x)$$

on the linear manifold of functions $u(x) \in C^{\alpha+2}(\bar{\Omega})$, $0 < \alpha < 1$ satisfying the boundary conditions (1.2) and (1.3). By Theorem 1.1, this operator is invertible and $D(\mathcal{L}_\Omega) \subset W_2^2(\Omega)$. Therefore \mathcal{L}_Ω^{-1} is a compact operator in $L_2(\Omega)$.

In addition, the inverse operator admits the integral representation (1.4). The self-adjointness of the operator \mathcal{L}_Ω^{-1} is easily obtained from (1.5). This follows from the symmetry of the kernel $G_P(x, y)$ of the integral operator (1.4). Consequently, the operator \mathcal{L}_Ω is self-adjoint as well. The proof of theorem 3.1 is complete. \square

Theorem 3.2. *The operator \mathcal{L}_Ω , corresponding to problem P, is positively definite, that is*

$$(\mathcal{L}_\Omega u, u) \geq \|u\|^2$$

for all $u(x) \in D(\mathcal{L}_\Omega)$.

Proof. It is easy to show that for any $u \in D(\mathcal{L}_\Omega)$ the following inequality

$$(\mathcal{L}_\Omega u, u)_{L_2(\Omega)} = \int_{\Omega} |\nabla u(y)|^2 dy \geq 0$$

holds. This fact, by virtue of the Friedrichs type inequalities, implies the assertion of theorem 3.2. \square

4. CONSTRUCTION OF EIGENFUNCTIONS OF THE PROBLEM P.

Since \mathcal{L}_Ω^{-1} is a compact operator, we find that its spectrum (and, consequently, the spectrum of the operator \mathcal{L}_Ω) can consist only of real eigenvalues of finite multiplicity. This operator has no associated functions. By Theorem 3.2 all the eigenvalues of the operator \mathcal{L}_Ω are positive. Consequently, they can be numbered in order of increasing

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots, k \in \mathbb{N}.$$

The eigenfunctions of the operator \mathcal{L}_Ω are found as solutions of the equation

$$-\Delta u(x) = \lambda u(x), x \in \Omega, \quad (4.1)$$

satisfying the boundary conditions (1.2) and (1.3).

For Eq. (4.1) consider two auxiliary problems:
Dirichlet problem

$$-\Delta v(x) = \lambda v(x), x \in \Omega; v(x) = 0, x \in \partial\Omega, \quad (4.2)$$

and Neumann problem

$$-\Delta w(x) = \lambda w(x), x \in \Omega; \frac{\partial w}{\partial n_x}(x) = 0, x \in \partial\Omega. \quad (4.3)$$

Both of the problems are self-adjoint, hence root subspaces consist only of eigenfunctions. All eigenvalues of the problems (except the first) are multiples.

The Dirichlet and Neumann problems possess the following symmetry properties of eigenfunctions.

Lemma 4.1. *All eigenfunctions of the Dirichlet problem (4.2) and of the Neumann problem (4.3) can be chosen so that they have one of the symmetry properties:*

$$v(x) + v(x^*) = 0, \quad (4.4)$$

or

$$v(x) - v(x^*) = 0. \quad (4.5)$$

Proof. We give the proof of lemma 4.1 only for the Dirichlet problem. Suppose that $\{v_k(x)\}_{k \in \mathbb{N}}$ is a system of all normalized eigenfunctions of the Dirichlet problem (4.2), and λ_k^D are corresponding eigenvalues. Denote

$$v_{kj}(x) = v_k(x) + (-1)^j v_k(x^*), j = 0, 1.$$

It may be that $v_{k0}(x) \equiv 0$ or $v_{k1}(x) \equiv 0$, but not simultaneously. Obviously, all these functions satisfy one of the symmetry conditions, that is, either condition (4.4) (if $j = 1$) or condition (4.5) (at $j = 0$). It is easy to see that $v_{kj}(x)$ are solutions of the Dirichlet problem

$$-\Delta v_{kj}(x) = \lambda_k^D v_{kj}(x), x \in \Omega;$$

$$v_{kj}(x) = 0, x \in \partial\Omega, j = 0, 1.$$

Therefore, if they are different from zero, then they are eigenfunctions.

Let us consider a system of functions

$$\{v_{k0}(x), v_{k1}(x)\}_{k \in \mathbb{N}}. \quad (4.6)$$

A part of these functions may be zero, but we not pay attention on this fact. We show that system (4.6) is complete in $L_2(\Omega)$. Indeed, suppose $g(x) \in L_2(\Omega)$ is orthogonal to all functions of system (4.6). Then for all $k \in \mathbb{N}$ we get

$$\begin{aligned} 0 = (v_{kj}, g) &= \int_{\Omega} v_{kj}(x) \overline{g(x)} dx = \int_{\Omega} \left[v_k(x) + (-1)^j v_k(x^*) \right] \overline{g(x)} dx \\ &= \int_{\Omega} v_k(x) \left[\overline{g(x)} + (-1)^j \overline{g(x^*)} \right] dx, j = 0, 1. \end{aligned}$$

Since the system of all eigenfunctions of the Dirichlet problem $\{v_k(x)\}_{k \in \mathbb{N}}$ is complete in $L_2(\Omega)$, then $g(x) \equiv 0$ for almost all $x \in \Omega$. This fact proves the completeness of system (4.6) in $L_2(\Omega)$. System (4.6) remains complete under removing zero functions from it. All non-zero functions of system (4.6) are the eigenfunctions of the Dirichlet problem (4.2). Since system (4.6) is complete in $L_2(\Omega)$, then the problem has no other (linearly independent) eigenfunctions. All system components (4.6) have the symmetry property (4.4) or (4.5). This proves Lemma 4.1 for the case of the Dirichlet problem.

For the Neumann problem (4.3) the proof is analogous. Lemma 4.1 is proved. \square

Remark 4.2. *The term "chosen so" means that the problem can have its eigenfunctions not satisfying (4.4) or (4.5). But in every root subspace one can choose other eigenfunctions, which will already satisfy (4.4) or (4.5).*

Theorem 4.3. *Eigenvalues of the problem P form two series. Accordingly, the system of eigenfunctions also consists of two series. The first series is the eigenfunctions of the Dirichlet problem (4.2) having the symmetry property (4.5). The second series is eigenfunctions of the Neumann problem (4.3) having the symmetry property (4.4).*

Proof. Let $\{v_{k0}(x)\}_{k \in \mathbb{N}}$ be a system of eigenfunctions of the Dirichlet problem (4.2) having the symmetry property (4.5), and let λ_{k0}^D be corresponding eigenvalues of the Dirichlet problem. Suppose $\{w_{k1}(x)\}_{k \in \mathbb{N}}$ is a system of eigenfunctions of the Neumann problem (4.3) having the symmetry property (4.4), λ_{k1}^N are corresponding eigenvalues of the Neumann problem. Obviously, the functions $v_{k0}(x)$ and $w_{k1}(x)$ are eigenfunctions of the non-local problem (4.1), (1.2), (1.3) corresponding eigenvalues λ_{k0}^D and λ_{k1}^N . We show that the system

$$\{v_{k0}(x), w_{k1}(x)\}_{k \in \mathbb{N}} \quad (4.7)$$

is complete in $L_2(\Omega)$. Then the problem P does not have other eigenfunctions.

We divide the space $L_2(\Omega)$ into a direct sum of two spaces: spaces $L_2^+(\Omega)$ of functions with the symmetry property (4.5) and spaces $L_2^-(\Omega)$ of functions with the symmetry property (4.4). By Lemma 4.1, the system $\{v_{k0}(x)\}_{k \in \mathbb{N}}$ is complete in $L_2^+(\Omega)$, and the system $\{w_{k1}(x)\}_{k \in \mathbb{N}}$ is complete in $L_2^-(\Omega)$. Consequently, system (4.7) is complete in $L_2(\Omega)$. Therefore, the nonlocal problem (4.1), (1.2), (1.3) has no eigenfunctions of another type. The proof of Theorem 4.3 is complete. \square

5. SOME SPECTRAL INEQUALITIES FOR THE EIGENVALUES OF THE PROBLEM P

Proof. (Theorem 2.1) From the results of Theorem 4.3 it follows that problem (1.1)-(1.3) has two series of eigenvalues. The first series coincides with the eigenvalues of

the Neumann problem $\lambda_{k1}^N(\Omega)$, and eigenfunctions are selected from the eigenfunctions of the Neumann problem with the symmetry property (4.4). The second series coincides with eigenvalues of the Dirichlet problem $\lambda_{k0}^D(\Omega)$, and eigenfunctions are selected from the eigenfunctions of the Dirichlet problem with the symmetry property (4.5). Therefore it is easy to see that $\lambda_1(\Omega) = \lambda_2^N(\Omega)$.

Also it is known [9] that if $\Omega \subset \mathbb{R}^n$ is such that the embedding $W_2^1(\Omega) \subset L_2(\Omega)$ is compact, then for all $k \in \mathbb{N}$ the inequality

$$\lambda_{k+1}^N(\Omega) < \lambda_k^D(\Omega)$$

is true. Consequently, $\lambda_1(\Omega) < \lambda_1^D(\Omega)$. This proves the theorem 2.1. \square

Proof. (Theorem 2.2) From the results of Theorem 2.1, we have that the first eigenvalue of non-local problem P coincides with the second eigenvalue of the Neumann problem: $\lambda_1(\Omega) = \lambda_2^N(\Omega)$. As follows from [13] (Szegő-Weinberger Theorem), the ball maximizes the second eigenvalue of the Neumann problem among all domains with equal measure. Inequality (2.1) is proved. \square

Proof. (Theorem 2.6) Consider the expression: $\frac{\lambda_1(\Omega)}{\lambda_1^D(\Omega)}$. It is known [13] (Faber-Krahn Theorem) that the ball minimizes the first eigenvalues of the Dirichlet problem among all domains with equal measure. And, by Theorem 2.2 we have inequality (2.1). Then expression (2.2) is obtained from the following chain of inequalities:

$$\frac{\lambda_1(\Omega)}{\lambda_1^D(\Omega)} \leq \frac{\lambda_1(B)}{\lambda_1^D(\Omega)} \leq \frac{\lambda_1(B)}{\lambda_1^D(B)}.$$

Inequality (2.3) is proved similarly. \square

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